Experiment 13: Periodic Motion

I. About the Experiment

Examples of oscillatory or periodic motion are familiar to everyone. An especially simple kind of periodic motion is represented by the behavior of the harmonic oscillator, which serves as an idealized model to represent the most important features of other periodic motions. The essential characteristics of the harmonic-oscillator model are discussed in Experiment 12. In this experiment you will examine two other characteristics of harmonic motion; damping and coupled oscillations.

II. Damping

Experience shows that when any real mechanical oscillating system is set into motion the oscillations eventually die out and the system comes to rest at its equilibrium position. The position of the mass as a function of time is not a simple sinusoidal function but rather a function with the general shape of Figure 1. This effect is due to the presence of damping forces (friction) in addition to the restoring force.

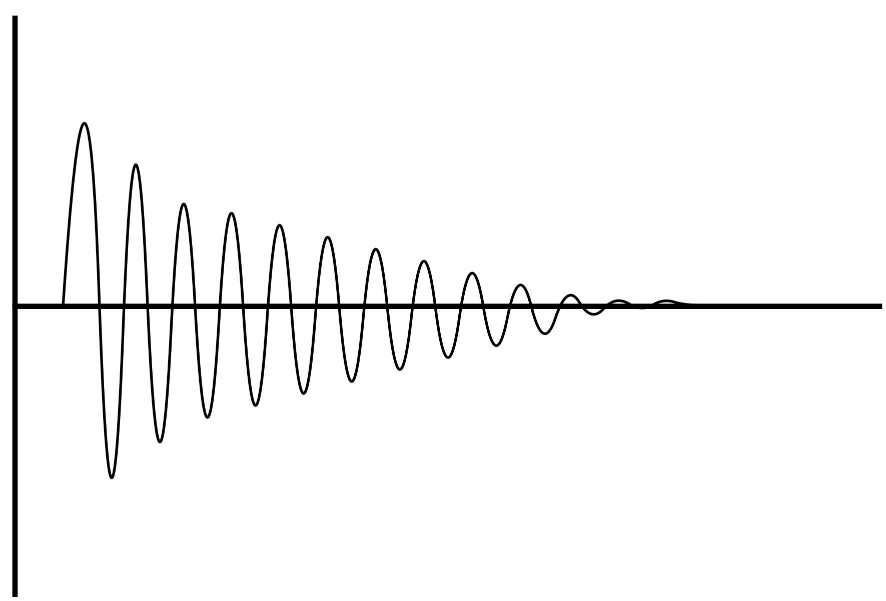

Figure 1

The effect of damping may be approximated by an additional force (F_d) that is proportional to velocity but opposite the direction of motion. This can be represented by

$$F_d = -bv = -b\frac{dx}{dt} \qquad \text{Eqn.1}$$

where b is called the damping constant, characterizing the strength of the damping force. Clearly, the rate at which oscillations die away depends on the magnitude of b; a large value of b means rapid decay. Including the damping force, Newton's second law gives

$$F_{net} = -kx - bv = -kx - b\frac{dx}{dt} = ma = m\frac{d^2x}{dt^2} \qquad \text{Eqn.2}$$

The general solution to Equation 2 for small damping is

$$x = A_0 e^{-(b/2m)t}\cos(\omega' t + \delta) \qquad \text{Eqn.3}$$

where

$$\omega_0 = \sqrt{k/m} \quad \text{and} \quad \omega' = \omega_0\sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2} \qquad \text{Eqn.4}$$

(small damping means $\left(\frac{b}{2m\omega_0}\right) < 1$).

From Equation 3 we see that the presence of the damping term leads to an exponential decay in the amplitude of the oscillation. Notice also that the frequency of oscillation $f' = \omega'/2\pi$ is less than the undamped oscillators frequency of oscillation $f = \omega_0/2\pi$. If the damping is very small, $\omega' \approx \omega_0 = \sqrt{k/m}$.

The phase constant (d) simply determines the value of t at which x is a maximum. If d is set equal to zero, the maximum displacement of the oscillating system will be A_0 at t = 0. From Equation 3 we see that the amplitude at any later time will be given by

$$A = A_0 e^{-\left(b/2m\right)t} = A_0 e^{-t/\tau} \qquad \text{Eqn.5}$$

where

$$\tau = \frac{2m}{b} \qquad \text{Eqn.6}$$

τ is called the **relaxation time** of the damped oscillator. Clearly the relaxation time is the time it takes the initial amplitude to decrease to $\left(\frac{1}{e}\right)$ of its original value.

Experimentally it is somewhat easier to determine the time it takes the amplitude of a damped oscillator's amplitude to decay to $\left(\frac{1}{2}\right)$ of its original value. The value of this time is represented by $\mathbf{t_{1/2}}$ and is called the **half life** of the damped oscillator**.**

There is a simple relationship between the half life and the relaxation time.

$$t_{1/2} = \tau \ln 2 \qquad \text{Eqn.7}$$

Oscillators are often characterized by their quality factor Q. Q is defined as ω' times the ratio of energy stored in the system to the energy dissipated in one cycle. Q is related to the relaxation time by:

$$Q = \omega' \frac{\text{Total Energy at begining of cycle}}{|\text{Energy lost during cycle}|} = \frac{2\pi}{T}\tau \qquad \text{Eqn.8}$$

where T' is the period of the damped oscillator. **An oscillator with a high Q loses very little energy per cycle.**

III. Coupled Oscillators

A simple example of coupled oscillators is shown in Figure 2. This is a system containing two equal masses coupled by identical springs. If one of the masses is displaced

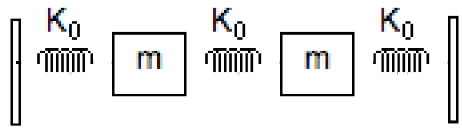

Figure 2

and then released, the resulting motion will not be sinusoidal. However, the system does have possible motions in which each mass moves sinusoidally. One possibility is for the two masses to move exactly in unison, so that the distance between them is constant. A little consideration shows that in this motion the center spring does not contribute to the restoring force on either mass, so the effective force constant is $2k_0$ and the effective mass is 2m. Thus, we expect the frequency of this motion to be given by

$$f_s = \frac{1}{T_s} = \frac{1}{2\pi}\sqrt{\frac{2k_0}{2m}} = \frac{1}{2\pi}\sqrt{\frac{k_0}{m}} \qquad \text{Eqn.10}$$

where the s subscript stands for symmetric. A second possibility is for the two masses to have exactly opposite motions. In this case the midpoint of the middle spring does not move; the motion of each mass is as though it were acted on one side by a spring of force constant k_0 and on the other side by a spring half as long. Halving the length doubles the force constant (Why?), so the total effective force constant for each mass is $3k_0$, and the corresponding frequency is

$$f_a = \frac{1}{T_a} = \frac{1}{2\pi}\sqrt{\frac{3k_0}{m}} \qquad \text{Eqn.11}$$

(the a subscript stands for anti-symmetric)

Any motion of a system of coupled oscillators such as this, in which all masses move sinusoidally with the same frequency, is called a normal mode of the system. Each normal mode has a characteristic frequency relationship between the motions of the various elements of the system.

IV. The Experimental Set-up

A. Single Dynamics Cart

A low friction dynamics cart on a horizontal dynamics cart track is attached at its ends to identical springs. Each spring has a force constant k_0; that is, to stretch either spring a distance x requires a force F = k_0x. In the equilibrium position both springs are stretched the same amount, so the total force is zero. Refer to Figure 3.

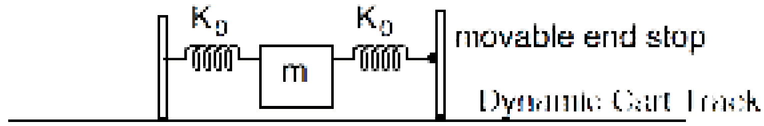

Figure 3

When the mass is displaced a distance x to the right of its equilibrium position, the force of the left spring increases by k_0x, whereas that of the right spring decreases by the same amount. The result is a net force to the left with magnitude $2k_0x$, so the undamped frequency of oscillation is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{2k_0}{m}} \qquad \text{Eqn.9}$$

B. Double Dynamics Carts

Two equal mass dynamics carts are placed on a horizontal dynamics cart track and are coupled together with identical springs. In the equilibrium position all springs are stretched the same amount, so that the total force both masses is zero. Refer to Figure 4.

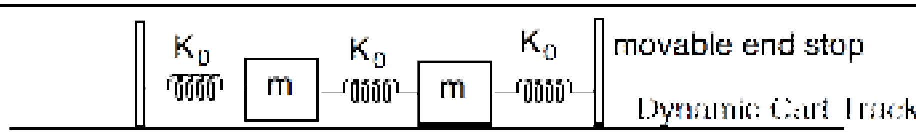

Figure 4

By displacing the carts appropriately the normal modes of oscillation can be excited.

V. Procedure

A. Spring Constant

1. Carefully match the mass of the two dynamics carts by using a pan balance and adding mass to one of the two carts.

Note: Step A1 may be done following step A4.

2. In order to measure the spring constant, set-up the track and one of the carts as shown in Figure 5.

3. Use the weight hanger with string attached to one of the dynamics carts. Note the equilibrium position of a reference point (usually the leading edge of the cart) on the dynamics cart.

4. Consider the initial conditions to be 0 weight and 0 displacement. Now add weights in 5 gm increments, up to a total of 25 gm, recording the total displacement of the reference line for each value of total weight. Make a table to record your data. (Note: Do not over stretch the spring, as it will be permanently deformed.)

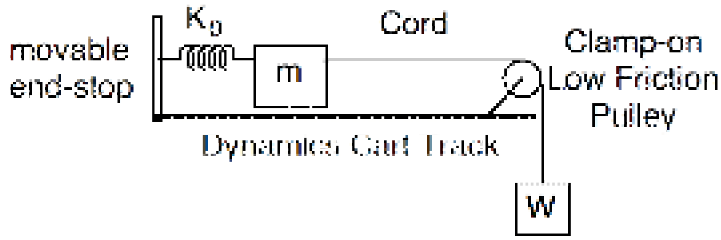

Figure 5

B. Simple Harmonic Motion

1. Remove the string from the dynamics cart and replace it with a spring identical to the first. Attach this spring to a movable end-stop and adjust the system to stretch each spring approximately 30 centimeters. Refer to figure 3.

2. Displace the dynamics cart 12 cm from the equilibrium position and measure the total time to make 5 complete cycles. Observe the motion, noting the energy transfer between the dynamics cart and the springs. Repeat the measurement with initial displacements of 14 and 16 cm.

C. Damping

1. Damping may be observed with the same experimental set-up. Carefully displace the cart 12 cm from its equilibrium position and release it from rest. Measure the time ($t_{1/2}$) and count the number of cycles (N) required for the amplitude to fall to half its original value. Note that T' will be given by $T' = \frac{t_{1/2}}{N}$.

D. Coupled Oscillators

1. Assemble the system shown in Figure 4. Adjust the movable end-stops such that each spring stretches approximately 30 cm.

2. Displace one dynamics cart, holding the other in place, and release both carts at once. Note the complex nature of the motion.

3. Displace both carts towards the center by the same amount and simultaneously release them. Does the motion now appear sinusoidal? (This is called the anti-symmetric mode.)

4. Measure and record the time for 5 complete oscillations.

5. Now displace the dynamics carts in the same direction by equal amounts and release them. Is the motion sinusoidal? (This is called the symmetric mode.)

6. Measure and record the time for at least 3 complete oscillations.

VI. Calculations and Analysis

1. Enter the data from Part V.A into a spreadsheet and use the computer to make a plot of spring extension (x) as a function of applied force (mg).

2. Determine the slope of the best fit straight line for your data points by using Excel's built in function LINEST(y_values, x_values). Since $x = \frac{1}{k} mg$ the slope of the line gives the reciprocal of the spring constant. Refer to Experiment 7 for more information about LINEST.

3. From your measured value for the mass of the dynamics cart and the value of k determined from the slope computed in VI. 2, compute the period T and compare this to your measured value of T. Refer to equation 9.

4. Using the data from part C, compute
   a) The relaxation time, τ.
   b) The damping constant, b.
   c) The quality factor, Q

   Be sure to show your calculations.

5. From the data of part D determine f_s and f_a. Compare these values to those predicted for f_s and f_a by Equations 10 and 11.

VII. Questions

1. How is the motion of the system shown in Figure (3) related to the motion of a simple pendulum?

2. Why is it desirable to use two springs in the set-up shown in Figure (3), rather than a single spring?

3. If a spring with spring constant k_0 is cut in half, what is the force constant of the resulting spring?

4. In this experiment the masses of the springs have been neglected. Will the effect of spring mass be to increase or decrease the frequency of oscillation? Explain.