I. INTRODUCTION

This paper discusses the application of Newtonian mechanics to determine whether a sled crossing a bridge or one going through a valley will reach the other side of the valley first (see Fig. 1). Although the winner is easily determined in the frictionless case, the inclusion of friction necessitates the solution for motion along a vertical curve. Constructing the valley out of only linear segments is unphysical. A massive object requires an infinite force to change its direction of motion by a finite angle as it moves from one linear segment to another at a different angle. Connecting these segments by small regions of high curvature necessitates either large normal forces and their associated large frictional forces when the center of curvature is above the path or fly-off problems when the center of curvature is below the path. In the former case, the energy loss results in changes in the speed of the object. One cannot simply let the radius of curvature go to zero and assume the speed of the object is unaffected. In the latter case the requirement that the object remain on the surface results in a lower limit for the radius of curvature, which is proportional to the square of the speed of the object.

II. THE OUTCOME OF THE RACE WITH NO FRICTION

It is reasonable to assume that the sled going into the valley is always in contact with the ground and that the normal force, that is, the support force that the ground exerts on the sled, is positive. This condition will be referred as the “no fly-off condition.”

In the frictionless case, conservation of energy tells us that when the two sleds approach the valley with the same horizontal speed, the sled on the bridge will retain this speed throughout its motion and the other sled will emerge from the valley with the same horizontal speed. Without friction the only force in the valley that has a horizontal component is the normal force. This force will cause the horizontal component of the sled’s velocity to increase as the sled descends and to decrease as the sled ascends out of the valley. Thus, the horizontal speed of the sled in the valley is always greater than or equal to that on the sled on the bridge and as a result the sled going through the valley will win the race. However, this simple argument is no longer applicable when there is friction.

III. INCLUSION OF FRICTION ON A STRAIGHT SLOPE

Before considering the effect of friction on the motion of a sled on a curved slope, it is useful to consider it for a straight slope. The frictional force, \( f \), is equal to the product of the coefficient of friction, \( \mu \), and the normal force, \( n \), and is opposite in direction to the velocity. Because there is no centripetal acceleration on a straight slope, the normal force is equal to the component of the weight normal to the slope. Hence, the acceleration along the slope is given by

\[
a = g \sin \theta - f/m = g(\sin \theta - \mu \cos \theta), \tag{1}
\]

where \( \theta \) is positive clockwise from the horizontal axis and \( m \) is the mass of the sled. Because

\[
a = dv/dt = (dv/ds)(ds/dt) = v dv/ds,
\]

where \( s \) is measured along the slope, Eq. (1) can be integrated to give the familiar result

\[
v^2 - v_0^2 = 2gs(\sin \theta - \mu \cos \theta) = 2g(\Delta y - \mu \Delta x)
\]

\[
= 2g \Delta x (\tan \theta - \mu), \tag{3}
\]

where \( v_0 \) is the speed at the beginning of the slope, \( x \) is measured horizontally in the direction of the motion, and \( y \) is measured vertically upward. Because the acceleration on a straight slope is constant, the time needed to transverse the horizontal distance, \( \Delta x \), is
\[ \Delta t = 2 \Delta x \sqrt{(v + v_\mu) \cos \theta}. \]  

(4)

For simplicity, we will construct the valley only out of circular arcs with no intervening straight regions. However, the earlier discussion is applicable to the sled moving across the bridge if we set \( \theta = 0 \).

**IV. INCLUSION OF FRICTION ON CIRCULAR ARCS**

On a circular arc the frictional force, \( f = \mu n \) and is directed as shown in Fig. 2. For a circular arc, \( n \) equals the component of the weight normal to the slope \( \pm \) the centripetal force, \( mv^2/r \), where \( r \) is the radius of the arc. The \( + \) sign for the frictional force is appropriate when the sled is moving on a concave (convex) surface. The acceleration along the arc is given by

\[ a = \frac{\eta g \sin \theta - f/m \eta g \cos \theta - \mu (g \cos \theta - \eta g v^2/r)}{m}, \]

where \( \eta = (+, -, -, +) \). In Fig. 2, the first and fourth arcs are convex surfaces with \( \eta = 1 \), while the second and third are concave and have \( \eta = -1 \). If we use Eq. (2) and the relation \( ds = rd\theta \) on an arc, the equation of motion can be rewritten as

\[ d(v_i^2/2gr)/d\theta = \eta_i \mu v^2_i/2gr - \mu \cos \theta + \eta_i \sin \theta. \]

(6)

For each arc, Eq. (6) is a linear, inhomogeneous, first-order differential equation whose solution is

\[ v_i^2/2gr = c_1 \exp(\eta_i \beta - a \sin \theta - b \cos \theta), \]

where \( \beta = 2 \mu \theta \), \( a = 6 \mu/(1 + 4 \mu^2) \), and \( b = 2(1 - 2 \mu^2)/(1 + 4 \mu^2) \), and the constant, \( c_1 \), is an integration constant to be determined, for example, \( c_1 = v_1^2/2gr + b \).

**V. THE ARC VALLEY**

We now evaluate the integration constants and tie the solutions together to find the speed at any point along the valley constructed from the four arcs. If the initial speed of the sled approaching the valley is \( v_0 \), then the square of the speed of the sled along the first arc is

\[ v_1^2/2gr = c_1 \exp(-a \sin \theta - b \cos \theta), \]

with \( c_1 = b + v_1^2/2gr \). If we equate \( v_1 \) at \( \theta = \theta_f \) to \( v_2 \) at \( \theta = -\theta_f \), that is, the junction of arcs 1 and 2, we obtain the speed along arcs 2 and 3:

\[ v_2^2/2gr = c_2 \exp(-b \sin \theta + b \cos \theta), \]

with \( c_2 = c_1 - 2(b \cos \theta + a \sin \theta) \exp(-\beta_f) \).

If we equate \( v_2^2/2gr = c_2 \exp(-\beta_f) \) to \( v_3 \) at \( \theta = -\theta_f \), the speed along arc 4 is

\[ v_2^2/2gr = c_3 \exp(-\beta_f) \exp(-\beta_f) \exp(-\beta_f) \exp(-\beta_f). \]

(10)

With \( c_4 = c_3 + 2(b \cos \theta - a \sin \theta) \exp(-\beta_f) \). If we set \( \mu = 0 \), \( \beta_f = 0 \), we obtain the familiar equations that result from conservation of energy.

On a curved slope the acceleration is not a constant and the time needed to move a horizontal distance is given by

\[ t = \int dx/v_i = \int ds/v = \int r d\theta/v. \]

(11)

In the frictionless case, the integral for the time for the sled to move along arcs 2 and 3 is an elliptic integral:

\[ t_{2,3} = \sqrt{\frac{\pi}{g}} \int \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \]

(12)

where \( \sin \phi = \sin(\theta/2)/k \) and \( k = \sqrt{1 - \cos \theta + v_0^2/4gr} \). The time needed to reach various points in the valley will be calculated numerically using Maple to evaluate the integrals.

**VI. THE OUTCOME OF THE RACE WITH FRICTION**

An inspection of the expressions for the speeds of the sleds given in Sec. V shows that three dimensionless parameters govern the outcome of the race: \( v_0^2/2gr \), \( \mu \), and \( \theta_f \). As a result of the no fly-off condition along the first arc, there are restrictions on the range of these parameters. For example, in the frictionless case, the no fly-off condition and conservation of energy result in the restriction

\[ 3 \cos \theta_f - 2 \geq v_0^2/2gr. \]

(13)

For \( \mu > 0 \), the no fly-off condition gives the weaker restriction \( 1 \geq v_0^2/2gr \). To ensure the no fly-off condition is satisfied, the normal was plotted for the first arc to see if it were positive. The plot revealed that some of the choices for \( v_0^2/2gr \), \( \mu \), and \( \theta_f \) were inconsistent with the no fly-off condition. We will stay well within the no fly-off region by taking \( v_0^2/2gr = 1/2 = \sin \theta_f \) and allowing \( \mu \) to increase from zero. If we use the expression for the speed on a horizontal line with friction Eq. (3) with \( \theta = 0 \) and the fact that the length of the bridge, \( x_B \), over the arc valley is \( 4 \pi \sin \theta_f \), we find that the value of \( \mu \) for which the sled on the bridge just makes it across the bridge is

\[ \mu_B = \frac{v_0^2}{2g}(8gr \sin \theta_f). \]

(14)

For \( v_0^2/2gr = 1/2 = \sin \theta_f \), \( \mu_B = 0.125 \).

The time needed for the sled on the bridge to cross the bridge is

\[ t_B = x_B/(v_0 + (v_0^2 - 2\mu_B x_B)^{1/2}). \]

(15)

The time needed to reach a particular horizontal position for the sled on the bridge, \( t_B \), and the time for the sled in the valley, \( t_v \), are divided by \( t_B \) to give the dimensionless time shown in the plots. The winner is the sled with the smallest dimensionless time at the end of the bridge, \( x/(r \sin \theta) = 4 \). For \( \mu = 0 \), Fig. 3(a) shows that the sled in the valley wins the race. Note that the tendency of the valley sled to lift off the slope in regions 1 and 4 results in lesser increases in its horizontal speed compared to the situation in...
region 2 and 3 where \( n_x \) and the resulting increases in the horizontal speed are significant. With no friction, the maximum speed is reached at the bottom of the valley and the curve is symmetric about the mid-point of the bridge.

Figure 3\( \sim b \) shows the case \( v^2_i / gr = 1/2 = \sin \theta_f \) and \( \mu = 0.06 \) for which the sled in the valley still wins, but crosses the finish line slightly slower than the sled on the bridge. Figure 3\( \sim c \) shows that when \( \mu = 0.1044 \), the race is a tie. The sled emerges from the valley so slowly that the faster moving sled on the bridge catches up and ties the race. Increasing \( \mu \) allows the sled on the bridge to win. The value \( \mu = 0.10515 \) results in the sled in the valley just barely able to leave the valley [see Fig. 3\( \sim d \)]. When \( \mu = \mu_B = 0.125 \), the sled on the bridge just barely finishes the race.

A comparison of the final speeds of the two sleds for various sets of \( v^2_i / gr \) and \( \theta_f \) shows that for \( \mu > 0 \), the final speed of the sled on the bridge is always greater than that of the sled in the valley. This condition can be exploited to choose a value of \( \mu \) that slows the sled in the valley enough to cause it to lose the race. The case \( v^2_i / gr = 1/2 = \sin \theta_f \) is typical in that there is a range of \( \mu \geq 0 \) in which the sled in the valley wins the race, a second much smaller range of \( \mu \) in which the sled in the valley finishes the race but moves so slowly that the other sled passes it just before the end, a third range in which only the sled on the bridge finishes, and then \( \mu > \mu_B \) for which neither sled makes it across the valley.

**VII. PROJECT FOR STUDENTS**

The kinetic and static coefficients friction are usually determined using a slider and an inclined plane. Devise an experiment to measure them using a concave surface of constant radius, for example, a wooden bowl or the inside of a tire. The angle of repose, whose tangent is the coefficient of static friction, could be just as easily measured as the maximum angle at which a slider would not move on the concave surface as on an incline. Instead of determining the coefficient of kinetic friction by deciding at which angle the slider would move at constant speed, release the slider on one side of the concave surface and measure the maximum angle that it reaches on the other side. Because the speed is zero at the initial and final angles, the coefficient of kinetic friction is determined by these two angles and Eq. (8) (with \( v_0 = 0 \) and \( \eta = -1 \)), for the speed along a concave arc. For example, a slider released at 90° will reach an angle of 60° if...
\( \mu = 0.115 \) and an angle of 75° if \( \mu = 0.050 \). Unless you work at small angles and use expansions, use Maple’s fsolve or similar software to solve the transcendental equation for the coefficient of kinetic friction. Because the position at which a slider momentarily stops can be determined more accurately than the angle of an incline at which a slider will move at constant speed, this method may be a more accurate way of determining the coefficient of kinetic friction. One advantage is that no one needs to catch the slider before it flies off the table.

You should consider the following questions. What is the best way to measure an angle on a circular surface? One way is to use a protractor, another is to measure the horizontal distance from the center of the surface and use \( x = r \sin \theta \). Another way is to measure the arc length and use \( s = r \theta \). Is it better to start at large initial angles near 90° or angles just above the angle of repose?

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2 For a typical example in a textbook see the two peak, ski problem in D. Halliday and R. Resnick, *Fundamentals of Physics*, extended 5th ed. (Wiley, New York, 1998), p. 184. In this problem if one connects the two slopes by an arc of negligible size and realizes that the speed will be reduced by a factor of \( \exp(-\mu \Delta \theta) \) in going through the arc, the resulting answer for the coefficient of friction is less than one half the value that one obtains by neglecting the effects of the connecting region. However, this result will not avoid the fact that the skier will encounter extremely large forces when the radius is negligibly small.
4 This result means that there is a lower limit for the radius \( v_0^2 / g \) of a arc segment connecting a horizontal slope to inclined slope.
5 One might recognize in this expression the solution of the “ice-ball” problem, \( \theta_f = \arcsin(2/3) \), the angle at which a mass starting from rest atop of an ice ball will fly off the surface. See for example, D. Halliday and R. Resnick, *Fundamentals of Physics*, extended 5th ed. (Wiley, New York, 1998), p. 180.